

# C++ LOPOR

## Library for Option Pricing and Operational Risk

## Manual<sup>©</sup>

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# Introduction

## 1. Introduction

The LOPOR library is an efficient library for option pricing and operational risk. It is user-friendly and easy to use in combination with other libraries. For this reason, and contrary to all libraries that I know, no special type of variable is defined. Only types defined in the standard library `std` are used.

This manual is composed of two parts.

The first part is devoted to present all tools (mainly statistical) necessary to solve problems in option pricing and operational risk. It is not a text book for operational risk and only a manual to use with the library. With these tools you are able to solve any problem in operational risk. For example see [[Vose2003](#)] which is readable and very interesting. Some other books less useful could be [[Marshall2001](#)] and [[Cruz2002](#)].

The second part is devoted to option pricing and could be considered as a text book with implementations. It cannot be considered as exhaustive and is still in expansion. The principle of option pricing is to not allow arbitrage, i.e. the possibility to earn money without risk. To understand this concept I advice you strongly to read the section [Simple binomial model](#) first. It present some fundamental points of option pricing, martingales and risk neutral considerations, through a very simple example. It is very useful to understand these concepts in this case before going to more complicated modelization.

Moreover I used many books to construct this library. It is worth to remark that each new book copy a great part of its content (99 %?) from old one. This is normal but what is less is:

1. They do do quote the reference of the original work

## 2. The copy is usually not well done

The last point is fundamental and explain why many books are simply not understandable and why it is usually better to read the original articles. I tried to reference the works who inspired me at each chapter. Moreover, there are some books that I used extensively: [[Baxter1996](#)], [[Joshi2003](#)], [[Wilmott2000](#)] for an introduction, [[James2001](#)], [[Rebonato2004a](#)], [[Brigo2001b](#)] for interest rate derivatives, and [[Clewlow1998](#)] and [Numerical Recipes](#) for implementation. I also used [[London2005](#)] but be aware that this book contains many errors in formulae, programs, and reasoning.

This library could have some bugs. If you find one, please send me an email. Moreover if you do not find a function which could be useful for you, or if you do not understand something, please send me an email: [Damien.Loison@physik.fu-berlin.de](mailto:Damien.Loison@physik.fu-berlin.de)

## 2. General Syntax

All the library uses the **LOPOR** namespace. You have two ways to include the library:

- You can include the "LOPOR.hpp" header:  
`#include "LOPOR.hpp"`
- Or you can include each class that you need. For example  
`#include "Exponential.hpp"`

## 3. Performance

"Our library is carefully made and extremely efficient ...". We could put the standard sentence but the truth is:

- The C++ is a slow language (compared to Fortran or C). Some attempts have been made to accelerate the C++. For example see the [[Blitz++](#)] library however we have not included it to keep the LOPOR library small enough and easily portable.
- The more friendly a language/library is, the less efficient it is because it cannot optimize all possible codes. In particular almost all functions of the distributions are virtual which allow users to create new distributions easily. But this slow down execution of programs.

In conclusion,

- If the performances are extremely important for you, you should do all your programs in C or in Fortran. You will gain until a factor 10 because of the language and also because you can optimize your code for your specific application.
- We tried to release an efficient C++ library keeping it as friendly-user as possible.

## 4. Exception Handling

The errors are managed through the [Error.hpp](#) class. An Error is thrown if there is a problem. The syntax to throw one error is:

```
#include "Error.hpp"
#include "Global.hpp"
LOPOR::Error("define the error" + LOPOR::c2s(value) + "what you want" );
```

**value** can be a double, integer, boolean, etc. We use the function [c2s\(\)](#) for "convert to string" defined in the class [Global.hpp](#). To catch the Error the program must look like:

```
// Example Error1.cpp download
//
#include "LOPOR.hpp"
using namespace LOPOR;

int main()
{
    try
    {
        Exponential exp;
        exp.setParameter(-2);
    }
    catch (const LOPOR::Error& error) { error.information( ); }
    return 0;
}
```

And the output of this program is:

```
Error: LOPOR::Exponential(-2)::setParameter( ) => The variable:-2 must be > 0
```

We can replace `error.information( );` by `std::cout << error.value << std::endl;`

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## References

- [**Ameur2002**] H. Ben Ameur, M. Breton, and P. L'Ecuyer, *A Dynamic Programming Procedure for Pricing American-Style Asian Options*, *Management Science*, **48** (2002) 625.
- [**Babbs2000**] S. Babbs, *Journal of Economic Dynamics & Control*, **24** (2000) 1499
- [**Barraquand1995**] J. Barraquand and D. Martineau, *Numerical valuation of high dimensional multivariate American securities*, *Journal of Financial and Quantitative Analysis*, **30** (1995) 383.
- [**Baxter1996**] M. Baxter, and A. Rennie, *Financial Calculus, an introduction to derivative pricing*, Cambridge university press, 1996
- [**BDT1990**] F. Black, E. Derman, and W. Toy, *A one-factor model of interest rates and its application to treasury bonds*, *Financial Analysis Journal*, **Jan-Feb** (1990) 33.
- [**Black1976**] F. Black, *The pricing of commodity contracts*, *Journal of Financial Economics*, **3** (1976) 167.
- [**BlackKarasinski1991**] F. Black, and P. Karasinski, *Bond and option pricing when short rate are lognormal*, *Financial Analysis Journal*, **July-August** (1991) 52.
- [**Blitz++**] *Object- Oriented Scientific Computing*, <http://www.oonumerics.org/blitz/>
- [**Boyle1977**] P. Boyle, *Journal of Financial Economics*, **4** (1977) 323
- [**Boyle1989**] P. Boyle, J. Evnine, and S. Gibbs, *Numerical evaluation of multivariate contingent claims*, *Review of Financial Studies*, **2** (1989) 241
- [**Boyle1997**] P. Boyle, M. Broadie, and P. Glasserman, *Journal of Economic Dynamics and Control*, **21** (1997) 1267

- [**Broadie1996**] M. Broadie and P. Glasserman, *Estimating security price derivatives by simulation*, *Management Science*, **42** (1996) 269.
- [**Broadie1997**] M. Broadie and P. Glasserman, *Pricing American-style securities using simulation*, *Journal of Economic Dynamics and control*, **21** (1997) 1323.
- [**Brigo2001a**] D. Brigo and F. [Mercurio](#), *Displaced and mixture diffusion for analytically-tractable smile model*, *Mathematical finance*, Bachelier Congress 2000, Geman, H., Madan, D. B., Pliska, S.R., Vorst, A.C.F., eds. Springer Finance, Springer, Heidelberg..
- [**Brigo2001b**] D. Brigo and F. [Mercurio](#), *Interest Rate Models: Theory and Practice*, Springer Finance, Heidelberg, 2001.
- [**Broyden1965**] C.G. Broyden, *Mathematics of Computation*, **19** (1965) 577
- [**Cairo1997**] M.C. Cairo, and B.L. Nelson, *Modeling and generating random vectors with arbitrary marginal distributions and correlation matrix* Technical Report, Department of Industrial and Management Sciences, Northwestern University, Evanston, IL, 1997.
- [**Cavalli-Sforza1996**] L.L. Cavalli-Sforza, P. Menozzi, A. Piazza, *The History and Geography of Human Genes : (Abridged paperback edition)* , Princeton University Press, 1996.
- [**Chancellor2000**] E. Chancellor, *Devil take the hindmost*, A plum book, penguin, 2000.
- [**Chen2001**] H. Chen, *Initialization for NORTA: generation of random vectors with specified marginals and correlations*, *Journal on computing*, **13** (2001) 312.
- [**Clewlow1998**] L. Clewlow and C. Strickland, *Implementing derivatives models*, ed. Wiley, 1998.
- [**Chourdakis2005**] K. Chourdakis, *The Derivatives-on-Line Pages, Lecture Notes for the course: Financial Derivatives*, <http://www.theponytail.net/DOL/lnet.htm>.
- [**Cobb2004**] B.R. Cobb and J.M. Charnes, *Approximating free exercise boundaries for American-style options using simulation and optimization*, *Proceedings of the 2004 Winter Simulation Conference*, Eds. R.G. Ingalls, M.D. Rosseti, J.S. Smith, and B.A. Peters.
- [**Cox1976**] J. Cox and S. Ross, *The valuation of options for alternative stochastic processes*, *Journal of Financial Economics*, **3** (1976) 145.
- [**Cox1985**] J. Cox and M. Rubinstein, *Option markets*, ed. Prentice Hall, 1985.
- [**Coyle1999**] L.N. Coyle and J.J. Yang, *Analysis of the SSAP Method for the Numerical Valuation of High-Dimensional Multivariate American Securities*, *Algorithmica*, **25** (1999) 75.
- [**Cruz2002**] M. G. Cruz, *Modeling, measuring and hedging operational risk*, ed. Wiley, 2002.
- [**Davison1997**] A.C. Davison, and D.V. Hinkley, *Bootstrap Methods and their Applications*, ed. Cambridge University Press, 1997.
- [**Derman1994**] E. [Derman](#) and I. Kani, *Riding on a smile*, *Risk*, **7** (1994) 32.
- [**Derman1995a**] E. [Derman](#), D. Ergener, and I. Kani, *Static options replication*, *Journal of derivatives*, **Summer** (1995) 78.
- [**Dupire1994**] B. Dupire, *Pricing with a smile*, *Risk*, **7** (1994) 18.
- [**Efron1993**] B. Efron, and R.J. Tibshirani, *An Introduction to the Bootstrap*, ed. Chapman&Hall, 1993.
- [**Evans2000**] E. Evans, N. Hastings, and B. Peacock, *Statistical Distributions*, ed. Wiley, Third edition, 2000.
- [**Fu2001**] M. Fu, S.B.Laprise, D.B. Madan, Y. Su, and R. Wu, *Pricing American Options: A*

*Comparison of Monte Carlo Simulation Approaches*, Journal of Computational Finance, **4** (2001) 39.

[**Garcia2003**] D. Garcia, *Convergence and Biases of Monte Carlo estimates of American option prices using a parametric exercise rule*, Journal of Economic Dynamics and Control, **27** (2003) 1855.

[**Gelman2000**] A. Gelman, J.B. Calin, H.S. Stern, and D.B. Rubin, *Bayesian Data Analysis*, ed. Chapman&Hall, 2000.

[**Ghosh2002**] S.Ghosh and S.G. Henderson, *Properties of the NORTA method in higher dimensions*, Proceedings of the 2002 Winter Simulation Conference. E. Yucesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes, eds. IEE, 2002

[**Glasserman1999**] P. Glasserman, P. Heiselberger, and P. Shahabuddin, *Mathematical Finance*, **9** (1999) 117

[**Glasserman2004**] P. Glasserman, *Monte Carlo methods in financial engineering*, Springer, 2004

[**GnuLibrary**] A free/open-source C and C++ library numerical library, <http://www.gnu.org/software/gsl/>

[**HoLee1986**] T.S.Y. Ho, and S.B. Lee, *Term structure movements and pricing interest rate contingent claims*, Journal of finance **41**(1986) 1011.

[**Hornick1989**] K. Hornick, M. Stinchcombe and H. White, *Multilayer feedforward networks are universal approximators*, Neural Networks **2**(1989) 359.

[**HullWhite1993**] J. Hull, and A. White, *One factor interest rate model and the valuation of interest rate derivative securities*, Journal of financial and quantitative analysis **28**(1993) 235.

[**HullWhite1994**] J. Hull, and A. White, *Numerical procedures for implementing term structure models I: single factor models*, Journal of derivatives **1**(1994) 21.

[**HullWhite1995**] J. Hull, and A. White, *A note on the model of Hull and White for pricing options on the term structure: response*, Journal of derivatives **5**(1995) 97.

[**Hull1997**] J. C. Hull, *Options, Futures, and Other Derivatives*, ed. Prentice-Hall, Third Ed., 1997.

[**Jackel2002**] P. Jackel. *Monte Carlo method in finance*, Wiley, 2002.

[**James2001**] J. James and N. Webber, *Interest rate modeling*, Wiley, 2001.

[**Jamshidian1989**] F. Jamshidian, *An exact bond option formula*, Journal of Finance, **XLIV** (1989) 205.

[**Johnson1994a**] N. Johnson, S. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*, ed. Wiley, Second edition, 1994.

[**Johnson1994b**] N. Johnson, S. Kotz, N. Balakrishnan, *Discrete Univariate Distributions*, ed. Wiley, Second edition, 1994.

[**Johnson1994c**] N. Johnson, S. Kotz, N. Balakrishnan, *Continuous Multivariate Distributions*, ed. Wiley, Second edition, 1994.

[**Joshi2003**] M. Joshi, *The Concepts and Practice of Mathematical Finance*, Cambridge University Press.

[**Joshi2004**] M. Joshi, *Design Patterns and Derivatives Pricing*, Cambridge University Press.

[**Kindleberger2000**] C.P. Kindleberger, *Manias, panics, and crashes*, Wiley, 4th edition, 2000.

[**Lalley2001**] S. Lalley, *Statistics and Mathematical Finance*, unpublished, accessible at <http://www.stat.uchicago.edu/~lalley/Courses/390/>.

[**Kleinert**] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, Wold Scientific, 3rd edition, 2004.

[**Lepage**] G.P. Lepage, *Journal of Computational Physics*, **27** (1978) 192

[**Levy1997**] E. Levy, *Exotic options: the state of the art*, Ed. L. Clewlow and C. Strickland, International Thomson Publishing, London, 1997.

[**Loison2000**] D. Loison, and P. Simon, [Phys. Rev. B \*\*61\*\* \(2000\) 6114](http://www.physik.fu-berlin.de/~loison/fast_algorithms/index.html), appendix

[**Loison2004**] D. Loison, C. Qin, K.D. Schotte, X.F. Jin, *Euro. Phys. J. B* **41** (2004) 395, accessible from [http://www.physik.fu-berlin.de/~loison/fast\\_algorithms/index.html](http://www.physik.fu-berlin.de/~loison/fast_algorithms/index.html).

[**Loison2004b**] D. Loison, [http://www.physik.fu-berlin.de/~loison/fast\\_algorithms/index.html](http://www.physik.fu-berlin.de/~loison/fast_algorithms/index.html).

[**London2005**] J. London, *Modeling derivatives in C++*, Wiley, 2005.

[**Longstaff2001**] F.A. Longstaff and E.S. Schwartz, *Valuing American options by simulation: a simple least-squares approach*, *The Review of Financial Studies*, **14**(2001) 113.

[**Marshall2001**] C. Marshall, *Measuring and managing operational risk in financial institutions*, Wiley, 2001.

[**Merton1976**] R.C. Merton, *Option pricing when underlying stock returns are discontinuous*, *Journal of Financial Economics*, **3**(1976) 125.

[**Newmat**] C++ Matrix library: <http://www.robertnz.net/>

[**Newton1997**] N.J. Newton, *Numerical methods in Finance*, editors L.C.G. Rogers, and D. Talay, Cambridge University Press, 1997.

[**Nielsen1999**] L.T. Nielsen, *Pricing and hedging of derivative securities*, Oxford University press, 1999.

[**Numerical Recipes**] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C++*, Cambridge University Press, second ed., 2002.

<http://www.library.cornell.edu/nr/cbookcpdf.html>

A must for numerical calculations, even if programs are simply a translation of FORTRAN.

[**Peterson1979**] R. Kronmal and A. Peterson, *American Statistician*, **33**(1979) 214.

[**Pizzi2002**] C. Pizzi, P. Pellizzari, *Monte Carlo Pricing of American Options Using Nonparametric Regression*, *Rendiconti per gli Studi Economici Quantitativi*, (2002) 75.

<http://econwpa.wustl.edu/eprints/fin/papers/0207/0207007.abs>

[**QuantLib**] A free/open-source C++ library for quantitative finance, <http://www.quantlib.org/>

[**Raymar1997**] S. Raymar, M. Zwecher, *A Monte Carlo valuation of American call options on the maximum of several stocks*, *Journal of Derivatives* **5**(1997) 7.

[**Rebonato2004a**] R. Rebonato, *Interest-rate option models*, ed. Wiley, Second edition, 2004.

[**Rebonato2004b**] R. Rebonato, *Volatility and correlation*, ed. Wiley, Second edition, 2004.

[**Sautter1996**] C. Sautter, *La Puissance Financiere du Japon*, Economica, 1996.

[**Schoenmakers1997**] J.G. Schoenmakers, and A.W. Heemink, *Journal of Computational Finance*, **1** (1979) 47.

[**Silvia1996**] D.S. Silvia, *Data Analysis: a Bayesian Tutorial*, ed. Oxford University Press, 1996.

[**Smith1965**] G.D. Smith, *Numerical Solution of Partial Differential Equations*, ed. Oxford

University Press, 1965.

Old but a perfect introduction to the finite differences.

**[Vapnik1999]** V.N. Vapnik, *The nature of statistical learning theory (statistics for engineering and information science)*, Springer Verlag, 1999, 2nd ed.

**[Sornette2004]** D. Sornette, *Why Stock Markets Crash : Critical Events in Complex Financial Systems*, ed. Princeton University Press, 2004.

**[Vose2003]** D. Vose, *Risk analysis, a Quantitative Guide*, ed. Wiley, second edition, 2003.

**[Wilmott2000]** P. Wilmott, J. Dewynne, and S. Howison, *Option pricing, mathematical models and computation*, ed. Oxford Financial Press, 2000.

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